Let $\Omega \subset \mathbb{R}^n$ be a smooth open bounded domain in $\mathbb{R}^n$. We call

$$u_{tt} = \Delta u, \tag{1}$$

the wave equation. To solve the equation, we give the Cauchy condition

$$u(x,0) = g(x), \quad u_t(x,0) = h(x), \qquad x \in \Omega,$$

as well as a lateral boundary condition such as the Dirichlet condition or the Neumann condition for the heat and Laplace equation.

1. Energy

Suppose that a solution $u(x,t)$ to the wave equation satisfying the Neumann condition that $\partial_\nu u = 0$ holds on $\partial\Omega$. Then, the energy

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2 + |\nabla u|^2 dx,$$

satisfies

$$\begin{aligned} E'(t) &= \int_\Omega u_t u_{tt} + \nabla u \cdot \nabla u_t dx = \int_\Omega u_t u_{tt} + \int_{\partial\Omega} u_\nu u_t dx - \int_\Omega (\Delta u) u_t dx \\ &= \int_\Omega u_t u_{tt} dx - \int_\Omega (\Delta u) u_t dx = \int_\Omega u_t (u_{tt} - \Delta u) dx = 0. \end{aligned}$$

Namely,

$$E(t) = E(0).$$

2. Separation of variables

Suppose that $u(x,t)$ satisfies $u_t t = c^2 u_{xx}$ for some $c > 0$ in $\{(x,t) : 0 \leqslant x \leqslant L, 0 \leqslant t\}$. Moreover, $u(0,t) = u(L,t) = 0$ holds for $t \geqslant 0$, and $u(x,0) = g(x)$, $u_t(x,0) = h(x)$ hold for $0 \leqslant x \leqslant L$.

Then, as like the Laplace and the heat equation, we consider a solution of the form $u(x,t) = v(x)w(t)$ with $v(0) = v(L) = 0$. Then, we can show $v(x) = A\sin(m\pi L^{-1}x)$ for $m \geqslant$. Let us denote it by v_m. Then, the corresponding $w_m(t)$ is $C\cos(m\pi L^{-1}ct) + D\sin(m\pi L^{-1}ct)$. Thus, we can obtain a solution

$$u(x,t) = \sum_{m=1}^{\infty} [a_m \cos(m\pi L^{-1}ct) + b_m \sin(m\pi L^{-1}ct)] \sin(m\pi L^{-1}x),$$

where

$$a_m = \frac{2}{L}\int_0^L g(x)\sin(m\pi L^{-1}x)dx, \qquad b_m = m\pi L^{-1}c\frac{2}{L}\int_0^L h(x)\sin(m\pi L^{-1}x)dx.$$

See section 5.3 of the textbook for more details.

3. 1D global Cauchy problem

Suppose that $u(x,t)$ satisfies $u_t t = c^2 u_{xx}$ for some $c > 0$ in $\{(x,t) : x \in \mathbb{R}, 0 \leqslant t\}$. Moreover, and $u(x,0) = g(x)$, $u_t(x,0) = h(x)$ hold for $x \in \mathbb{R}$, where g, h are smooth. Then, the d'Alembert formula yields the unique solution

$$u(x,t) = \frac{1}{2}\big[g(x+ct) + g(x-ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)dy. \tag{2}$$

See section 5.4 of the textbook.

Here, we observe an interesting fact that the formula give the unique solution without an infinity boundary data. We recall that the heat equation must need an infinity boundary data to have the unique solution. This phenomenon can be explained by the propagation speed.

For example, we consider the Cauchy data $g(x) = 0$, $h(x) > 0$ for $|x| < \delta$, and $h(x) = 0$ for $|x| \geqslant \delta$. Then, we can observe that $u(x,t) = 0$ for $|x| \geqslant ct + \delta$ and $u(x,t) > 0$ for $|x| < ct + \delta$. Hence, the wave equation has the finite propagation speed.

Suppose that $u(x,t)$ satisfies $u_t t = c^2 u_{xx} + f(x,t)$ for some $c > 0$ in $\{(x,t) : x \in \mathbb{R}, 0 \leqslant t\}$, where f is smooth. Moreover, and $u(x,0) = u_t(x,0) = 0$ hold for $x \in \mathbb{R}$. Then, the Duhamel's methods yields the unique solution

$$u(x,t) = \frac{1}{2c}\int_0^t ds \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy. \tag{3}$$

See section 5.4 of the textbook.

4. Backward uniqueness for the wave equation and the heat equation

Given a solution $u(x,t)$ to the wave equation $u_{tt} = \Delta u$, we consider $\tau = -t$. Then, we have $u_{\tau\tau} = \Delta u$. Therefore, we can solve the wave equation in the backward time.

However, it is hard to solve the heat equation in the backward time. We show the backward uniqueness of the solution to the heat equation.

Suppose that $u_t = \Delta u$ and $v_t = \Delta v$ hold in Q_T. Moreover, we have $u = v$ on $\partial\Omega \times [0, T]$ and $u(x, T) = v(x, T)$ on $x \in \Omega$. Then, we define a new solution $w = u - v$ to the heat equation and the energy

$$E(t) = \int w^2 dx.$$

Then, we have

$$E' = -2 \int |\nabla w|^2 dx, \qquad E'' = 4 \int |\Delta w|^2 dx.$$

Therefore, the Hölder inequality shows

$$|E'|^2 = 4\left(\int |\nabla w|^2 dx \right)^2 = 4\left(\int w(\Delta w) dx \right)^2 \leqslant 4\left(\int w dx \right)^2 \left(\int |\Delta w|^2 dx \right)^2 = EE''. \tag{4}$$

Assume that $E(t) > 0$ on some interval $I = [a, b) \in [0, T]$ and $E(b) = 0$. Remind that we have $E(T) = 0$ and thus $E(b) = 0$ hold for some $b \leqslant T$. Then, $f = \log E$ satisfies

$$f'' = (E''E - E'^2)E^{-2} \geqslant 0$$

on I. Namely, $f'(t) \geqslant f'(a)$, and thus

$$f(t) - f(a) \geqslant \int_a^t f'(a) ds = (t - a) f'(a).$$

However, $\lim_{t \to b} f(t) = -\infty$ since $E(b) = 0$. Contradiction.

Therefore, we have $E(t) = 0$ on $t \in [0, T]$, namely the solution is unique.